The class of opportunity filters and the preference revealed by a path independent choice function

Somdeb Lahiri
School of Economic and Business Sciences
University of Witwatersrand at Johannesburg

Key words
opportunity filters,
preference relations,
choice functions,
path independence.

JEL classification C79, D71.

Abstract

The purpose of this paper is to replicate the theory developed by Gekker (2001), without using any monotonicity assumption. We however retain a non-triviality assumption implicit in Gekker (2001), which says that there is at least one opportunity set which is preferred to the no-choice situation. In addition we require our preference relation on opportunity sets to be transitive (as in Gekker, 2001), reflexive and satisfy an assumption called minimal comparability, which requires every opportunity set to be comparable with the null set. We also show that there exists a preference relation on the power set of the set of alternatives, revealed by a path independent choice function, which satisfies all the properties that we require of a binary relation to satisfy on the power set of the set of alternatives.

1_ Introduction

The study of the problem of choosing from non-empty subsets of a given non-empty finite set, occupies a central place in economic theory. The non-empty subsets from which choices are made are called opportunity sets. There is a growing literature, which studies preferences over opportunity sets. A comprehensive survey of the traditional developments can be found in Barbera, Bossert and Pattanaik (2002).

In recent papers, Vannucci (2000) and Gekker (2001) provide analyses of preferences over opportunity sets in terms of the set of filters that may be defined relative to the preference. A filter is a non-empty collection of opportunity sets, such that, if an opportunity set belongs to the filter, then so do all others, which are at least as good as the given one. We call such filters *opportunity filters.* Gekker (2001), requires an assumption called monotonicity, which says that any opportunity set is at least as good as any other that it contains. This, as recent studies by Van Hees (1999) and Lahiri (2001) point out, is often an untenable assumption. The appearance of an undesirable alternative in an opportunity set, may render it less acceptable than the opportunity set that does not contain such alternatives (or even the situation of not having any alternatives to choose from, if the undesirable alternative happens to be particularly abhorring!), if one is not guaranteed that in the subsequent act of choice, one is free to discard whatever alternatives one currently dislikes. Further, it is often the case that the choice of an opportunity set is delegated to an individual who is different from the individual who would be choosing from the chosen opportunity set, as for instance in sequential decision making, although the final act of choice affects the individual responsible for choosing the opportunity set. Thus if an opportunity set contains an alternative which its chooser intensely dislikes, while the one responsible for choosing from the opportunity set finds the alternative highly desirable, then anticipating what the final outcome would be, the decision maker responsible for choosing opportunity sets would be better off by selecting the opportunity set which does not contain that alternative, while being otherwise the same.

The purpose of this paper is to replicate the theory developed by Gekker (2001), without using any monotonicity assumption. We however retain a non-triviality assumption implicit in Gekker (2001) which says that there is at least one opportunity set, which is preferred to the no-choice situation. In addition we require our preference relation on opportunity sets to be transitive (as in Gekker, 2001), reflexive and satisfy an assumption called minimal comparability, which requires every opportunity set to be comparable with the null set. Unlike Gekker (2001), reflexivity does not follow from any of the assumptions we make. Reflexivity is definitely a less demanding assumption than monotonicity is. Further, the analysis in Gekker (2001) does not require our minimal comparability assumption either. However, comparability of all opportunity sets with the no choice situation is not a very demanding requirement. Along with transitivity, it simply implies that an opportunity set which is at least as good as the no choice situation, is also at least as good as any opportunity set which is no better than the no choice situation.

In a final section of this paper, we show that a preference relation on the power set of the set of alternatives, revealed by a path independent choice function (Arrow, 1963 and Plott, 1973), and which is similar to the one defined by Johnson and Dean (2001) on the range of a path independent choice function, satisfies all the properties that we require of a binary relation to satisfy on the power set of the set of alternatives. This shows that the context of our analysis is non-vacuous.

2_ The Model

Let X be a non-empty finite set of alternatives. Let P(X) denote the power set of X, and [X] the set of all non-empty subsets of X. Let $\geq$ be a reflexive and transitive binary relation on P(X) and let $>$, $\sim$ denote its asymmetric and symmetric parts respectively.

We assume that $\geq$ satisfies the following two axioms:

- Axiom N (Non-trivial): there exists $A \in [X]$: $A > \phi$;
- Axiom MC (Minimal Comparability): for all $A \in P(X)$, either $A \geq \phi$ or $\phi \geq A$.

A non-empty subset Σ of [X] is said to be an opportunity filter with respect to $\geq$ if the following condition is satisfied: $[B\in\Sigma, A \geq B]$ implies $[A\in\Sigma]$.

Let $F(\geq)$ denote the set of opportunity filters with respect to $\geq$.

Let $\Delta(\geq) = \{A\in[X]/A > \phi\}$.

By Axiom N, $\Delta(\geq) \neq \phi$. By the transitivity of $\geq$, $[B\in\Delta(\geq), A \geq B]$ implies $[A\in\Delta(\geq)]$. Thus, $\Delta(\geq)\in F(\geq)$.

3_ The set of opportunity filters with respect to a binary relation on P(X)

Lemma 1: Let $\Gamma(\geq)$ be a subset of [X] satisfying the following properties:

a. for all $A, B\in P(X)$: $[A\in\Gamma(\geq), B\notin\Gamma(\geq)] \rightarrow A > B$;
b. for all $A, B\in P(X)$: $[A\notin\Gamma(\geq), B\notin\Gamma(\geq)] \rightarrow A \sim B$.

Then $\Gamma(\geq) = \Delta(\geq)$.

Proof: Let $A\in\Delta(\geq)$. Thus $A > \phi$. Since $\Gamma(\geq) \subset [X]$, $\phi\notin\Gamma(\geq)$. If $A\notin\Gamma(\geq)$, then by (b), $A \sim \phi$, contradicting $A > \phi$. Thus, $A\in\Gamma(\geq)$. Hence, $\Delta(\geq) \subset \Gamma(\geq)$.

Since $\Delta(\geq) \neq \phi$, it follows that $\Gamma(\geq) \neq \phi$.

Now, suppose $A\in\Gamma(\geq)$. Since $\Gamma(\geq) \subset [X]$, $\phi\notin\Gamma(\geq)$. By (a), $A > \phi$. Thus, $A\in\Delta(\geq)$. Hence, $\Gamma(\geq) \subset \Delta(\geq)$.

Thus, $\Gamma(\geq) = \Delta(\geq)$. Q.E.D.

Lemma 2: i. Let $\Gamma_1, \Gamma_2\in F(\geq)$. Then $\Gamma_1 \cup \Gamma_2\in F(\geq)$;
ii. If $\Gamma\in F(\geq)$, then $\Gamma \subset \Delta(\geq)$;
iii. If $\Gamma_1, \Gamma_2\in F(\geq)$ and $\Gamma_1 \cap \Gamma_2 \neq \phi$, then $\Gamma_1 \cap \Gamma_2\in F(\geq)$.

Proof: (i) and (iii) are immediate from the definition of an opportunity filter with respect to $\geq$. Hence let us prove (ii). Let $\Gamma\in F(\geq)$ and towards a contradiction suppose $\Gamma \not\subset \Delta(\geq)$. Thus, there exists $A\in\Gamma\backslash\Delta(\geq)$. By Axiom MC, $\phi \geq A$. Since $\Gamma\in F(\geq)$, $[A\in\Gamma$ and $\phi \geq A]$ implies $[\phi\in\Gamma]$, contradicting $\Gamma \subset [X]$. Thus, $\Gamma \subset \Delta(\geq)$. Q.E.D.

Let Σ be any non-empty subset of $\Delta(\geq)$. Let (Σ) denote the smallest opportunity filter with respect to $\geq$ containing Σ. (Σ) is called the opportunity filter with respect to $\geq$ containing Σ.

Clearly $(\Sigma) = \{A \in [X]/A \geq B$ for some $B \in \Sigma\}$. Denoting the latter set by Ω, it is immediate from the definition of an opportunity filter that Ω is a subset of any opportunity filter containing Σ. Further, $\Sigma \subset \Omega$ implies $\Omega \neq \phi$. Also $\phi \notin \Omega$, since $B > \phi$, whenever $B \in \Sigma$. By the transitivity of $\geq$, $[B \in \Omega, A \geq B]$ implies $[A \in \Omega]$. Thus, $\Omega \in F(\geq)$. Since, Ω is a subset of any opportunity filter containing Σ, it must be the case that $\Omega = (\Sigma)$. Note however that the requirement "let Σ be any non-empty subset of $\Delta(\geq)$" cannot be replaced by "let Σ be any non-empty subset of [X]", as the following example reveals:

Exemple: Let $X = \{1, 2\}$ and suppose $\{1, 2\} \sim \{1\} > \{2\} \sim \phi$. Let $\Sigma = \{\{2\}\}$. Since $\{2\} \sim \phi$, and any opportunity filter is required to be a non-empty subset of [X], there is no opportunity filter containing $\{2\}$.

Any non-empty subset of $\Delta(\geq)$ is called a base for an opportunity filter with respect to $\geq$.

Lemma 3: Let Σ and Γ be bases for opportunity filters with respect to $\geq$.

i. $\Sigma \subset \Gamma$ implies $(\Sigma) \subset (\Gamma)$;
ii. if Σ is an opportunity filter then $\Sigma = (\Sigma)$;
iii. $(\Sigma) = \cup\{(\Gamma)/\Gamma \subset \Sigma$ and Γ is a base for an opportunity filter with respect to $\geq\}$;
iv. $F(\geq) = \{\Sigma \subset [X]/\Sigma$ is a base for an opportunity filter with respect to $\geq$ and $\Sigma = [\Sigma]\}$.

Proof: (i) and (ii) are immediate. (iii) follows from (i) and that Σ itself is a base for an opportunity filter with respect to $\geq$. (iv) follows from the fact that any opportunity filter is also a base for itself. Q.E.D.

Proposition 1: Let $B > \phi$. Then, $A \geq B$ if and only if [for all $\Phi \in F(\geq)$: $B \in \Phi$ implies $A \in \Phi$].

Proof: Suppose $A \geq B > \phi$. Then clearly $\Phi \in F(\geq)$ and $B \in \Phi$ implies $A \in \Phi$.

Now suppose $B > \phi$ and not $(A \geq B)$. Consider $(\{B\}) = \{C \in [X] / C \geq B\}$. Clearly $A \notin (\{B\}) \in F(\geq)$, although $B \in (\{B\})$. This proves the proposition. Q.E.D.

Proposition 2: Let $A, B > \phi$. Then $A \geq B$ if and only if $(\{A\}) \subset (\{B\})$.

Proof: If $A \geq B > \phi$, then $A \in (\{B\}) \in F(\geq)$. Thus $(\{A\}) \subset (\{B\})$.

Now suppose $A, B > \phi$ and $(\{A\}) \subset (\{B\})$. Since $A \in (\{A\})$, we get $A \in (\{B\})$. Thus, $A \geq B$. Q.E.D.

Suppose there exists $\Gamma, \Sigma \in F(\geq)$, such that $\Gamma \not\subset \Sigma$ and $\Sigma \not\subset \Gamma$. Hence there exists $A \in \Gamma \backslash \Sigma$ and $B \in \Sigma \backslash \Gamma$. Thus: [not $(B \geq A)$ and not $(A \geq B)$]. Further, if there exists $A, B \in \Delta(\geq)$ such that [not $(B \geq A)$ and not $(A \geq B)$], then $B \notin (\{A\})$ and $A \notin (\{B\})$. Thus we have proved the following:

Proposition 3: The following statements are equivalent:

i. for all $A, B \in \Delta(\geq)$: either $A \geq B$ or $B \geq A$;
ii. for all $\Gamma, \Sigma \in F(\geq)$: either $\Gamma \subset \Sigma$ or $\Sigma \subset \Gamma$.

4_ The binary relation on P(X) defined by a non-empty subset of [X]

Let Θ denote the collection of all non-empty subsets of [X] and let G be any non-empty subset of Θ. Given $A, B \in P(X)$, let $A \geq_G B$ if and only if [for all $\Phi \in G$: $B \in \Phi$ implies $A \in \Phi$]. Let $>_G$ and $\sim_G$ denote the asymmetric and symmetric parts of $\geq_G$. Let $\Lambda(G) = \cup\{\Phi / \Phi \in G\}$.

$A \geq_G B$, $B \geq_G C$ implies:

i. for all $\Phi \in G$: $B \in \Phi$ implies $A \in \Phi$;
ii. for all $\Phi \in G$: $C \in \Phi$ implies $B \in \Phi$.

Hence, for all $\Phi \in G$: $C \in \Phi$ implies $A \in \Phi$.

Thus $A \geq_G C$. Thus, $\geq_G$ is transitive.

Clearly, if $A \in \Lambda(G)$, then $A >_G \phi$. Thus, $\geq_G$ satisfies Axiom N. Further, $A \in \Lambda(G)$, $B \in P(X) \backslash \Lambda(G)$ implies $A >_G \phi \sim_G B$. Thus, $\geq_G$ satisfies Axiom MC.

Proposition 4: $F(\geq_G) = \{\Gamma \subset \Lambda(G) / \text{(i) } \Gamma \neq \phi; \text{(ii) } [B \in \Gamma, A \geq_G B] \text{ implies } A \in \Gamma\} \equiv F$.

Proof : Let $\Gamma \in F(\geq_G)$. Suppose, there exists $A \in \Gamma$ such that $A \notin \Lambda$. Thus $A \sim_G \phi$ implies $\phi \in \Gamma$ contradicting $\Gamma \in F(\geq_G)$. Thus, $\Gamma \in F(\geq_G)$ implies $\Gamma \subset \Lambda(G)$. Since $\Gamma \in F(\geq_G)$, $[B \in \Gamma, A \geq_G B]$ implies $A \in \Gamma$. Thus, $\Gamma \in F$. Thus, $F(\geq_G) \subset F$.

Now, suppose $\Gamma \in F$. Since $\Gamma \subset \Lambda(G)$, $\phi \notin \Gamma$.

Further, $\Gamma \neq \phi$ and $[B \in \Gamma, A \geq_G B]$ implies $A \in \Gamma$. Thus, $\Gamma \in F(\geq_G)$. Thus, $F \subset F(\geq_G)$. Thus, $F = F(\geq_G)$. Q.E.D.

5_ The Preference Revealed by a path independent choice function

A function $C: [X] \rightarrow [X]$ such that for all $A \in [X]$: $C(A) \subset A$, is called a choice function. A choice function C is said to be Path Independent (PI) if for all $A, B \in [X]$: $C(A \cup B) = C(C(A) \cup C(B))$.

A choice function C is said to be Indempotent if for all $A \in [X]$: $C(A) = C(C(A))$.

Lemma 4: Let C be a PI choice function. Then C is Idempotent.

Proof: Let $A \in [X]$. Since C is PI, $C(A) = C(A \cup C(A)) = C(C(A) \cup C(C(A)))$ (by PI) $= C(C(A))$. Q.E.D.

Lemma 5: A choice function C is PI if and only if $A, B \in [X]$: $C(A \cup B) = C(A \cup C(B))$.

Proof: Suppose C is PI and let $A, B \in [X]$. Then, $A, B \in [X]$: $C(A \cup C(B)) = C(C(A) \cup C(C(B)))$ (by PI) $= C(C(A) \cup C(B))$ (by Lemma 4) $= C(A \cup B)$ (by PI).

Now suppose that for all A, B∈[X]: C(A∪B) = C(A∪C(B)). Let A, B∈[X]. Thus, A, B∈[X]: C(A∪B) = C(A∪C(B)) = C(C(A)∪C(B)). Thus, C is PI. Q.E.D.

Let C be a PI choice function. Let ≥ be a binary relation on P(X) defined as follows:

i. for all A∈[X] and B∈P(X), let A ≥ B if and only if C(A∪B) = C(A);
ii. ϕ ≥ ϕ. Let > and ~ denote the asymmetric and symmetric parts of ≥ respectively.

Clearly, A > ϕ, whenever A∈[X]. Thus ≥ satisfies both Axioms N and MC. Further, ≥ is reflexive. Let A, B, D∈P(X), with A ≥ B ≥ D. Since every element of [X] is preferred to ϕ, we can without loss of generality assume A, B, D∈[X].

A ≥ B implies C(A∪B) = C(A) and B ≥ D implies C(B∪D) = C(B). Now, C(A∪D) = C(C(A)∪D) (by Lemma 5) = C(C(A∪B)∪D) = C(A∪B∪D) (by Lemma 5) = C(A∪C(B∪D)) (by Lemma 5) = C(A∪B) = C(A). Thus, A ≥ D.

Hence, we have proved the following:

Proposition 5: Let C be a PI choice function. Let ≥ be a binary relation on P(X) defined as follows:

i. for all A∈[X] and B∈P(X), let A ≥ B if and only if C(A∪B) = C(A);
ii. ϕ ≥ ϕ.

Let > and ~ denote the asymmetric and symmetric parts of ≥ respectively. Then, ≥ is reflexive, transitive and satisfies Axioms N and MC.

6_Conclusion

It is perhaps true, that hidden in the results obtained above are some basic results on order filters of any finite preordered set (not necessarily a power set). The purpose of this paper is not to claim originality as a contribution to the theory of order filters, but to characterize the concept of flexibility that is embodied in a preference relation on the set of opportunity sets, in terms of a "decision theoretically" plausible class of filters. Hence we call all such filters opportunity filters. Thus,

for instance in the case of a partially ordered set, the empty set may be included as an element of a filter.
In our definition of opportunity filters, each of which is required to be a non-empty collection of non-empty subsets of a universal set, such a possibility is ruled out. Since the empty set corresponds to the theoretically uninteresting case of a no choice situation, including it in an opportunity filter would lead to little if any value addition to the results reported here.

While forfeiting a possible opportunity to appeal to or replicate an existing result in the theory of order filters may be interpreted as a lack of mathematical finesse, it definitely does not impair the quality of our results and presentation, which are mainly decision theoretic. The fact that in our chosen context our results can be established in a self contained manner, is in itself a vindication of the framework and mode of analysis.

References

ARROW, K. J. *Social choice and individual values*. 2nd Edition. New Haven: Yale University Press, 1963.

BARBERA, S.; BOSSERT, W.; PATTANAIK, P. K. Ranking sets of objects. In: BARBERA, Salvador; HAMMOND, Peter J.; SEIDL, Christian. (Eds.). *Handbook of utility theory*. Kluwer Academic Publishers, 2002. v. 2.

GEKKER, R. On the axiomatic approach to freedom as opportunity: a general characterization result. *Mathematical Social Sciences,* v. 42, p. 169-177, 2001.

JOHNSON, M.; DEAN, R. Locally complete path independent choice functions and their lattices. *Mathematical Social Sciences*, v. 42, p. 53-87, 2001.

LAHIRI, S. *Freedom of optimal choice*. Wits University, 2001. Mimeografado.

PLOTT, C. *Path independence, rationality and social choice*. Econometrica, v. 41, p. 1075-1091, 1973.

VAN HEES, M. Freedom of choice and diversity of options: some difficulties. In: SWART, H. de (Ed.). *Logic, game theory and social choice*. Proceedings of the International Conference, LGS'99, p. 491-503, 1999.

VANNUCCI, S. *Freedom of choice and filtral opportunity rankings:* representation and aggregation. Italy: University of Siena, 2000. Mimeografado.

I would like to thank two anonymous referees of this journal for drawing my attention to several historical and mathematical inaccuracies in an earlier version of this paper. I would like to thank one of them in particular, for making valuable suggestions about Section 5. Needless to say that the responsibility for remaining errors is mine.

Author's e-mail
lahiris@sebs.wits.ac.za
lahiri@webmail.co.za