A TUTORIAL INTRODUCTION
TO NONLINEAR DYNAMICS IN ECONOMICS

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ABSTRACT

The last few years have witnessed a great interest in nonlinear dynamics. This fascinating subject has enabled both theoreticians and practitioners in many fields to handle and analyse "classical" problems in a new and promising way. Although some of the main ideas in nonlinear dynamics are not necessarily complicated, the average researcher is not sufficiently acquainted with the basic concepts and terminology and this usually constitutes a difficulty which very few overcome. The main objective of this paper is to introduce the nonspecialist reader to some key ideas which have been recently used in the analysis of dynamical systems and time series in economics. In particular, special attention will be given to nonlinear dynamics, phase space reconstruction, attractors, dynamical invariants and the diagnosis of chaotic and nonlinear dynamics from time series. A number of references are provided for further reading.

1 INTRODUCTION

In the last three decades great attention has been devoted to the study of nonlinear dynamics. With very few exceptions, most of the first papers published in this area were written either by physicists or mathematicians. Originally, the primary concern was to develop new mathematical tools to understand observed nonlinear phenomena which could not be analysed using the already available concepts developed for linear systems.

As far as experimentation goes, two different scenarios were common. In one situation, certain dynamical phenomena displayed by a real system (usually a mathematical model or a laboratory "set up") needed to be studied and for this new approaches were required. Another situation arose when new mathematical concepts and numerical algorithms, developed for the analysis of nonlinear phenomena, had to be tested, preferably in the context of a real situation. In either case the researcher was confronted with a problem in which he or she had both data and algorithms. Combining these, it was usually possible, especially in cases when some dynamical property of the system was known *a priori* (as it often happens in laboratory experiments), to verify if the algorithms were adequate and if they yielded the expected results.

In the last years it has become apparent that, since all that is required apart from such algorithms are the data, the application of such new tools to the analysis of a much wider class of problems is not only possible but also desirable. In many of these new situations, however, a number of difficulties arise due to the fact that the benefits of a controlled environment attained in a laboratory are no longer present. Such difficulties are usually caused by short time series of noisy data.

The new tools developed as a consequence of the so-called *chaos advent* have gained popularity among practitioners in many different fields such as engineering (Mess, Sparrow, 1987), medicine (Goldberger *et al.*, 1990), ecology (Schaffer, 1985) and biology (Hassell *et al.*, 1991), to list a few. Economics, of course, is no exception to this rule (Boldrin, 1988; Brock, Malliaris, 1989; Brock *et al.*, 1991; Rosser, 1991, Boldrin, 1992; Jaditz, Sayers, 1993; Peel, Speight, 1994).

The interest in nonlinear dynamics in economics can be verified by the rapidly growing number of related papers. The basic appeal for considering nonlinear dynamics in the analysis of economic data is that many observed phenomena can now be explained based on concepts such as dynamical bifurcations, catastrophe theory and chaos (Day, 1985).[3] Moreover, if a time series is represented using appropriate coordinates, the result is, of course, no longer a time series but rather a geometrical object which can be studied using tools which are not standard in time series analysis. These issues will be addressed in sections 4, 5 and 6 where references will be provided for further reading.

These appealing benefits prompted a number of authors to apply the new tools and ideas from nonlinear dynamics to economics. Mathematically-based introductions to the subject have been presented by (Scheink-

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3 Other apparent advantages will be mentioned in the following sections.

man, 1990; Brock, Dechert, 1991) whereas a more readable and informal approach has been followed in (Baumol, Benhabib, 1989). However, in the latter paper the vital issue of quantification of nonlinear dynamics has not been addressed.

The aim of this paper is to provide a "non-mathematical" introduction to nonlinear dynamics and chaos in economics addressing the issues of quantification and diagnosis of nonlinearities and chaos. As a thorough introduction to the subject would be impractical, only some of the issues which are believed to be of greater relevance to economists will be presented. Hopefully, the cited references will meet any further needs of the more curious reader.

The paper is organised as follows. Section 2 provides a historical review of dynamics in economics. Section 3 will address the question of why should we be concerned with nonlinear dynamics and chaos in economics. Section 4 briefly reviews the fundamental problem of phase-space reconstruction from time series. Section 5 introduces the concepts of dynamical attractors and section 6 reviews two dynamical invariants used to quantify chaotic attractors. The diagnosis of nonlinearity and (eventually) of chaos from a time series is addressed in section 7. The main points of the paper are reviewed in section 8 where a number of references are provided for further reading.

2 DYNAMICS IN ECONOMICS – A HISTORICAL PERSPECTIVE

Traditionally, the majority of economists' theoretical research focusses on static phenomena and, when the dynamic methodology is used, attention tends to concentrate on problems with time paths that converge to stationary states. Interest in formal mathematical models that study persistent oscillatory dynamic behavior started in the 1930s. The discussion of persistent oscillations and *chaos* in economic models is a more recent phenomenon. In order to evaluate the impact of these new ideas on the theoretical field of economic dynamics it is necessary to understand which were the prevalent research paradigms in those times.

The origins of economists' interest in complex dynamics can be traced to the enormous literature on business cycles. Due to the complexity of business cycles and the many differences between them, their main features and causes have long been a matter of debate. At the beginning some economists related their causes with natural forces, others to psychological factors, and still others to the workings of the monetary and banking system. Toward the end of the nineteenth century the focus began to shift towards industry and employment phenomena, especially to the great fluctuations that characterised the capital goods industries. This vast nonmathematical

literature was composed of a large number of models trying to provide a set of conditions sufficient to generate oscillatory behavior similar to that observed in the economy. In most of the cases, however, these models were vague and did not lend themselves to empirical validation.

Starting in the 1930s this situation began to change with the contributions of several economists who used difference and differential equation models to generate deterministic time paths of economic variables (Frisch, 1933; Samuelson, 1939; Ezekiel, 1938; Domar, 1946). Ezekiel's simple "cobweb model" of market price determination is probably the first contact of economics students with dynamics (Chiang, 1982). Samuelson's famous multiplier-accelerator model – another well-known example – uses the framework of Keynesian macroeconomic theory. The latter explores the dynamic process of the national income when the accelerator principle works together with a Keynesian multiplier. It has the following structure:

$$Y_t = C_t + I_t + G_0,$$

$$C_t = \gamma Y_{t-1} \qquad\qquad (0 < \gamma < 1),$$

$$I_t = \alpha (C_t - C_{t-1}) \qquad (\alpha > 0),$$

where the national income, Y_t, includes the rate of consumption, C_t, the rate of investment, I_t, and government expenses, G_0 (exogenous). Consumption in period t is proportional to income in period $(t - 1)$. Induced investment is a function of the trend in the rate of consumption. Substituting the second equation in the third leaves investment as a function of the previous period's rate of growth of national income (the acceleration principle).

After proper substitutions in the first equation the model can be condensed into a single equation. Samuelson's equation is a nonhomogeneous second-order linear difference equation with fixed coefficients:

$$Y_t - \gamma (1 + \alpha) Y_{t-1} + \alpha \gamma Y_{t-2} = G_0.$$

Depending upon the particular values of the characteristic roots of a specific model the time path of the national income may be oscillatory or nonoscillatory, convergent or explosive. The different results obtained combining these possibilities are, in a sense, qualitatively equivalent no matter what the order of the difference equation is, as long as it is linear. As Baumol says,

> "this range of possible time path configurations simply was not sufficiently rich for the economists' purposes, since in reality time paths are often more complicated and many oscillations do not seem

either to explode or dampen toward disappearance"
(Baumol, Benhabib, 1989).

A solution to the above problem seemed to be the appearance of nonlinear models which played an important role in modelling economic dynamics starting in the 1940s (Kaldor, 1940; Hicks, 1950; Goodwin, 1951). Based on real economic issues, not just mathematical formulations, these authors showed that such models can generate solutions of a stable limit cycle type toward which all possible time paths of the dependent variable converge. It was Goodwin who spent the most efforts toward an endogenous explanation of economic fluctuations using nonlinear relations in the analysis of dynamic economic processes (Goodwin, 1982).

> *"By the 1960s, however, the profession had largely switched to the linear approach making use of Slutzky's (1927) observation that stable low order stochastic difference equations could generate cyclic processes that mimicked actual business cycles"*
> (Scheinkman, 1990).

Since then on, and until the 1980s, what can be observed is the dominance of the linear stochastic difference equation approach – at least in the area of business cycle modelling. Two reasons are pointed out to explain this tendency: in the first place, it was noted that nonlinear systems did not reproduce some aspects of actual economic time series; on the other hand, the competing models seemed to capture some of the features of aggregate economic time series even with low order autoregressive processes. So, the analysis of dynamic systems in economics has often been based on linear – or linearised – models.

In the 1970s new analytical tools to study nonlinear dynamic phenomena were developed in the natural sciences and these contributions produced important spillovers to economics. These developments led to the phenomenon of *chaos* in which a dynamic nonlinear mechanism that is very simple and deterministic yields a time path so complicated that appears random. By now it is well established that deterministic systems can generate dynamics that are extremely irregular (May, 1976).

Besides showing that a simple deterministic nonlinear relationship – such as a first order nonlinear difference equation – can yield an extremely complex time path, chaos theory addresses the general question of

> *"the (in)stability of deterministic, nonlinear dynamic systems which are able to produce complex*

motions of such nature that they are sometimes seemingly random" (Nijkamp, Reggiani, 1995).

It should be stressed that such systems show trajectories that sometimes display sharp qualitative changes in behavior, that are extremely sensitive to microscopic changes in the values of the parameters and, to mention only the most interesting features, they incorporate the characteristic that small uncertainties may grow exponentially (although all time paths are bounded) property known as *sensitivity to initial conditions* (Nijkamp, Reggiani, 1995).

Currently, economists' interest in chaotic dynamics is largely based upon the fact that most observed economic time series seem to have a certain degree of randomness and, consequently, such time series are difficult to predict. Some implications of chaos for economic modelling are pointed out briefly in the next section. In section 8 will be presented a concise exploratory overview of some contributions of this approach in several areas of economics.

3 NONLINEAR DYNAMICS: RENEWED HOPES

The use of linear models for modelling and forecasting time series in the decades before the last can be justified considering that there were theoretical and computational limitations to the use of nonlinear models. Besides, the estimation and analysis of linear models had still a long way to go. However, such models had two main drawbacks when used to model economic data, namely

i) deterministic linear models were unable to account for the uncertainties in observed data;

ii) these models could not reproduce sustained oscillations.

In order to overcome these two difficulties, it was necessary to include in linear models stochastic variables and also exogenous inputs. Hence, the extra stochastic variables provided some uncertainty whilst the exogenous shocks would provide 'random excitation' which would ensure that the model output would oscillate. As pointed out by Day,

> *"Traditionally, economic irregularity has been explained by the superimposition of random shocks on what is (usually) assumed to be a stable deterministic linear process"* (Day, 1985).

A simple example of this has been illustrated in Figure 1 . As can be seen, the temporal behaviour and the autocorrelation function of a 'random' and a chaotic time series might look very much the same. In other words, chaotic models can be considered as candidates when sustained oscillations with some degree of randomness are to be analysed. On the other hand, the well defined deterministic structure underlying the data is clearly revealed when the time series is reconstructed in phase-space. This will be discussed further in the following section.

Figure 1
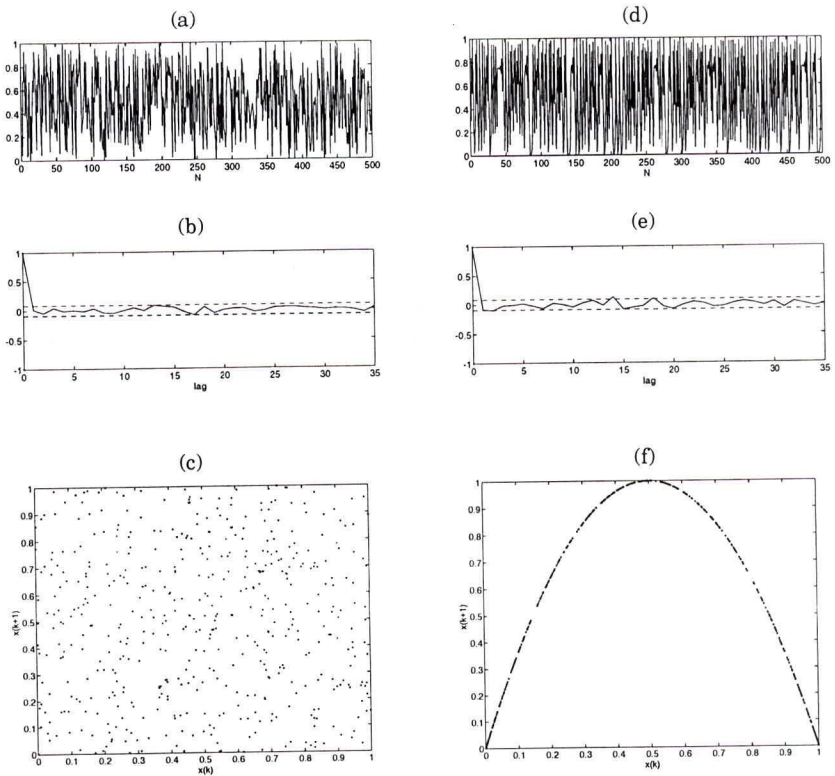
Chaotic time series and their respective autocorrelation functions may appear random. Phase-space reconstructions, however, can reveal some hidden structure.
(a) Time series produced by a random number generator, (b) the corresponding autocorrelation function and (c) phase space reconstruction. (d) Time series produced by the map $y(k) = 4.0\,[1 - y(k-1)]\,y(k-1)$, (e) the autocorrelation function and (f) the clearly revealed attractor in the reconstructed phase space.

With this in mind, it comes as no surprise that nonlinear dynamics and chaos have caught the attention of many researchers. Nonlinearities can produce sustained oscillations without any external shocks. Moreover, chaotic systems exhibit complicated oscillation patterns with some degree of impredictability. Yet another feature of nonlinear systems is that qualitatively different behaviors are produced as one (or more) parameter(s) of the system evolves in time.

Therefore, the introduction of nonlinearities in economic models reduces the importance of the role played by the external accidents ('shocks') in the explanation of the observed fluctuations of economic variables. Furthermore, different policy conclusions can be obtained from the same structural model; what makes the difference between one case or another is the value of some key parameter of the nonlinear model. For some range of values in the parameter space one can obtain stable solutions typical of 'classical' economics; for other set of the parameter values, the same structure produces the kind of 'unstable'[4] results characterising most of Keynesian economics or more complex solutions such as cycles and even chaos (Grandmont, 1985). This situation is very different from the results obtained from linear macroeconomic models where different policy views require different theoretical structures.

The use in econometric analysis of simple linear models with stochastic disturbances may be – in some cases – inappropriate and even misleading. In such cases, the use of nonlinear models would be more adequate. Once the linearity restriction is abandoned it is possible to think of a single theoretical structure that produces different policy recommendations depending upon the specific values of some key parameters. These developments suggest that instead of the usual discussions about alternative models the real issue may be the empirical research necessary to estimate the values of the parameters in a unique nonlinear model.

As a consequence of this tendency a different line of research is producing an abundant literature in which the efforts are directed towards testing the available time series searching for evidence of nonlinearities and chaos in the underlying dynamic processes (Brock *et al.*, 1991). This important issue will be addressed in section 7.

As M. Boldrin from UCLA clearly states,

> *"it is probably not unfair to say that nonlinear dynamics has not had a major impact on the development of modern economic theory. In fact, one*

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4 By unstable it is meant locally unstable, while global stability is implied. Chaotic dynamics are typical examples of this situation.

may even be tempted to add that, until very recently, it was either an unfamiliar tool for the mathematical economist or one whose implications were often disregarded as irrelevant to the purposes of the research. Dynamical systems theory appeared for a while in the background of the studies on the stability of the 'tâtonnement' process and on optimal growth and turnpike, but never really got on the stage" (Boldrin, 1988).

In the last few years, however, we witness a revival of interest on the study of nonlinear techniques as a result of the new ideas related to chaotic dynamics and the endogenously generated economic fluctuations explained as a deterministic phenomenon (Scheinkman, 1990). What we find in the most recent literature on the subject is that many authors try to find the 'roads to chaos in economics'[5].

The objective of these efforts is to produce theoretical models with 'reasonable' economic hypothesis (especially individual maximising behaviour and competitive market clearing) and parameter values, that predict cycles and chaos as logical outcomes. In view of the built-in stabilising mechanisms operating in the typical capitalist economies, it was realised that, in order to obtain chaotic dynamics, these mechanisms must be 'shut down'. According to this, W. A. Brock presents a list of economic factors that can explain chaos in dynamic economic evolution:

i) the intensity of people impatience, *i.e.*, the extent to which the economic agents behave myopically in relation to the future;

ii) the absence of the usual properties of concave tastes and technologies;

iii) the lack or imperfection of capital markets that permit borrowing-lending against expected future returns;

iv) nonequilibrium systems;

v) the existence of externalities in preferences or technologies;

vi) exogenous 'forcing functions' such as dynamics of technological change (Brock, 1988).

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5 This expression originated in the classical article by Leo Kadanoff, from Chicago, who studied the transition to turbulence in hydrodynamical systems (Kadanoff, 1983).

Finally, it is common to find in the literature dealing with nonlinearity and chaos the recognition that what is really missing are efforts to test the validity of the theoretical models by comparing the existing data with their predictions.

4 PHASE-SPACE RECONSTRUCTION

One technique used in the analysis of nonlinear dynamical systems is to plot a steady-state trajectory of a system in the phase-space. Thus if $y(t)$ is a trajectory of a given system this can be achieved by plotting $\dot{y}(t)$ *against* $y(t)$. As discussed below, in many practical situations only one variable is measured. In such cases an alternative procedure is to plot $y(t - T_p)$ against $y(t)$ where T_p is a time lag. Such variables define the so-called pseudo-phase plane and the choice of T_p is largely a matter of graphical representation of the data and is not a critical issue. This procedure is motivated by the fact that the embedded trajectories represented in the pseudo-phase plane have properties similar to those of the original attractor represented in the phase plane (Moon, 1987). This is discussed in more detail in what follows.

4.1 DYNAMICAL EMBEDDINGS

An nth-order dynamical system can be represented as a set of n first-order differential or difference equations each governed by a *state variable*. The global system thus has n time variables $\left[y_1, y_2, ..., y_n\right]$ and the solution of such a system can be thought of as n time series.

In a sense, the n time series mentioned above are obtained from the original nth-order system by decomposition. Also, given the n times series it is possible to recover the original n-dimensional solution by taking each state variable to be a coordinate of a 'reconstruction space' and to represent each time series in such a space. Thus n time series can be used to *compose* or *reconstruct* the system solution or trajectory. This is illustrated in Figure 2.

A difficulty encountered in practice with this approach is that n, the order of the system, is seldom known and even when an accurate estimate of this variable exists the number of measurements will not be as large as n. Take for instance the atmosphere which is usually thought of as a high-order system, but monitoring and weather forecasting stations only measure a very limited number of variables of this system.

Figure 2
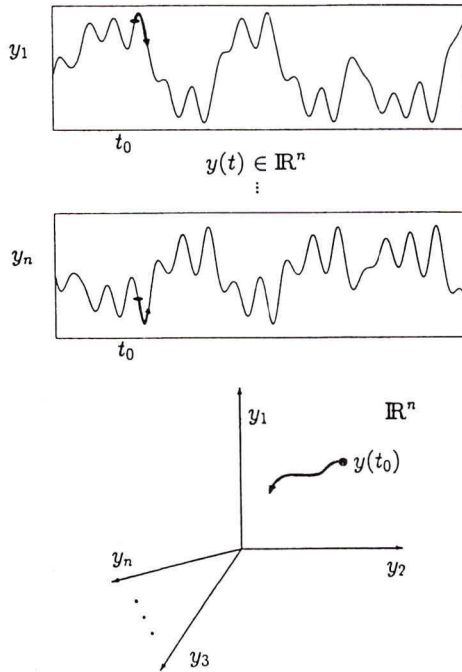
The n time series defined by the state variables of an nth-order dynamical system can be used to compose the trajetory in state space.

This can be described in a more mathematical way by conside-ring the action of a *measuring* function $h(y): \mathbb{R}^n \to \mathbb{R}$ which operates on the entire state or phase space but which yields just a scalar which is called the *measured variable*. The question which naturally arises at this stage is the following: given $f: \mathbb{R}^n \to \mathbb{R}^n$ and $h(y): \mathbb{R}^n \to \mathbb{R}$ is it possible to reconstruct a trajectory or solution of f from the scalar measurement $h(y)$?

Fortunately, it turns out that this question has an affirmative answer if certain requirements are met (Takens, 1980; Packard *et al.*, 1980; Sauer *et al.*, 1991). Thus *embedology* is concerned with how to reconstruct the phase space of a dynamical system of order n from a limited set of measurements q where $q < n$, and more often than not $q = 1$. In other words, the objective is to reconstruct the phase space of a system from a single time series. The resulting phase space is usually referred to as *embedded phase space*, *embedding space* or just *embedding*.

Another question which should be addressed is: why should we be concerned in reconstructing the trajectories of a dynamical system? Briefly, if time series are used to reconstruct the phase space of dynamical systems via embedding techniques, it is possible to use results from differential geometry and topology to analyse the resulting attractors which are *geometrical objects* in the reconstructed space as can be seen in Figure 1f. On the other hand, a purely random time series has no structure and therefore no geometry is revealed in the phase-space, see Figure 1c.

An important point to mention is that both the reconstructed and the original attractors are equivalent from a topological point of view, or in other words, they are diffeomorphic.

The practical consequences of this are obvious. No matter how complex a dynamical system might be, even if only one variable of such a system is measured, it is possible to reconstruct the original phase space via embedding techniques. It is also, at least in principle, possible to estimate quantitative invariants of the original attractor, such as the fractal dimension and Lyapunov exponents, directly from the reconstructed attractor which is topologically equivalent to the original one. These ideas are illustrated in Figure 3.

Figure 3
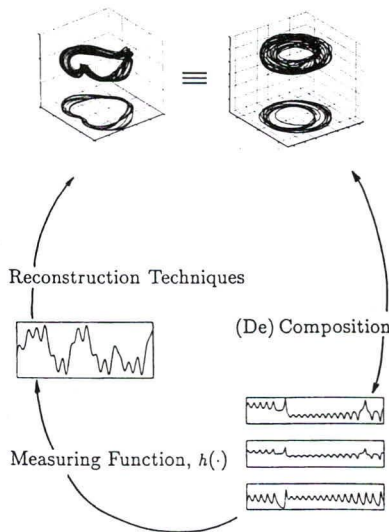
In many practical situations the number of measured variables is limited. Embedding techniques enable the reconstruction of the state space even from a single measurement. The reconstructed (or embedded) and the original state spaces are equivalent.

A convenient but by no means unique way of reconstructing phase spaces from scalar measurements is achieved by using *delay coordinates* (Packard *et al.*, 1980; Takens, 1980; Sauer *et al.*, 1991). Other coordinates include the *singular value* (Broomhead, King, 1986) and *derivatives* (Gouesbet, Maquet, 1992). A framework for the comparison of several reconstructions has been developed in (Casdagli *et al.*, 1991) and three of the most common methods have been studied in (Gibson *et al.*, 1992).

A *delay vector* has the following form

$$y(k) = [\, y(k) \; y(k \, - \, \tau) \; ... \; y(k \, - \, (d_e \, - \, 1) \, \tau \,) \,]^{\,T} \qquad (1)$$

where d_e is the *embedding dimension* and τ is the *delay time*. Clearly, $y(k)$ can be represented as a point in the d_e-dimensional *embedding space*. Takens has shown that embeddings with $d_e \, > \, 2n$ will be faithful generically so that there is a smooth map $f_t : I\!R^{d_e} \to I\!R$ such that (Takens, 1980)

$$y(k \, + \, T) = f_T \, (y(k)) \qquad (2)$$

for all integers k, and where the *forecasting time* T and τ are also assumed to be integers. A consequence of Taken's theorem is that the attractor reconstructed in $I\!R_e^d$ is diffeomorphic to the original attractor in state space and therefore the former retains dynamical and topological characteristics of the latter.

In the case of delay reconstructions, the choice of the reconstruction parameters, d_e and τ, is of the greatest importance since such parameters strongly affect the quality of the embedded space. The selection of d_e has been investigated in (Kennel *et al.*, 1992). The choice of the delay time has been discussed in (Kember, Fowler, 1993; Aguirre, 1995). Many authors have suggested that in some applications it is more meaningful to estimate these parameters simultaneously; this is tantamount to estimating the embedding window defined as $(d_e \, - \, 1)\, \tau$ (Martineire *et al.*, 1992). Some of these methods have recently been compared in (Rosenstein *et al.*, 1994). Dynamical reconstructions from nonuniformly sampled data has been addressed in (Breedon, Packard, 1992).

Taken's theorem gives sufficient conditions for equation (2) to hold, that is, in order to be able to infer dynamical invariants of the original system from the time series of a single variable, however no indication is given as to how to estimate the map f_T. A number of papers have been devoted to this goal, see (Abarbanel *et al.*, 1990; Casdagli, 1991) for piecewise linear techniques and (Kadtke *et al.*, 1993; Aguirre, Billings, 1995) for global

nonlinear models. It goes without saying that such models can be used not only for reconstructing the original dynamics but are also quite useful for forecasting.

5 ATTRACTORS

One of the nice features of phase-space reconstruction is that an embedded time series can be thought of as a geometrical object. If a deterministic and stable system 'operates' for a sufficiently long time without external shocks, it will reach the so-called (dynamical) *steady-state* regime which does *not* imply that the system is still[6]. In phase space this corresponds to the trajectories of the system falling onto a particular 'object' which is called the *attractor*. Asymptotically stable linear systems excited by constant inputs have point attractors which have dimension zero and correspond to a constant time series. Nonlinear systems, on the other hand, usually display a wealth of possible attractors. To which attractor the system will finally settle depends on the system itself and also on the initial conditions.

It should be pointed out that the *shape* and *dimension* of the attractors in phase space are directly related to the complexity of the dynamics of the respective time series. Thus, simple low dimensional attractors correspond to simple time series dynamics whereas more complex time series lie on attractors with higher dimension. This is illustrated in Figure 4.

Figure 5 shows a time series of the monthly number of people registered as unemployed in the former West Germany for the period January 1948-May 1980. The steep increase in the number of unemployed in the beginning can be due to war effect, and the sudden increase in the 1970s coincides with the world energy crisis and the world recession in general (Subba-Rao, Gabr, 1984). Reconstructing the phase space for these data has suggested that there are several different "attractors" underlying the data. Such "attractors", of course, have different geometries in the reconstruction space and this corresponds to the system having diverse dynamical properties over different periods of time, a fact which would hardly come as a surprise.

The most common attractors are the *point attractors* (dimension zero), *limit cycles* (dimension one) and *tori* (dimension two). Another type of attractor which has recently deserved a great deal of attention are the so-called *strange* or *chaotic attractors* which are *fractal* objects. Such attractors will be introduced below.

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6   Similarly, a time series does not have to be constant to be (statistically) stationary.

Figure 4
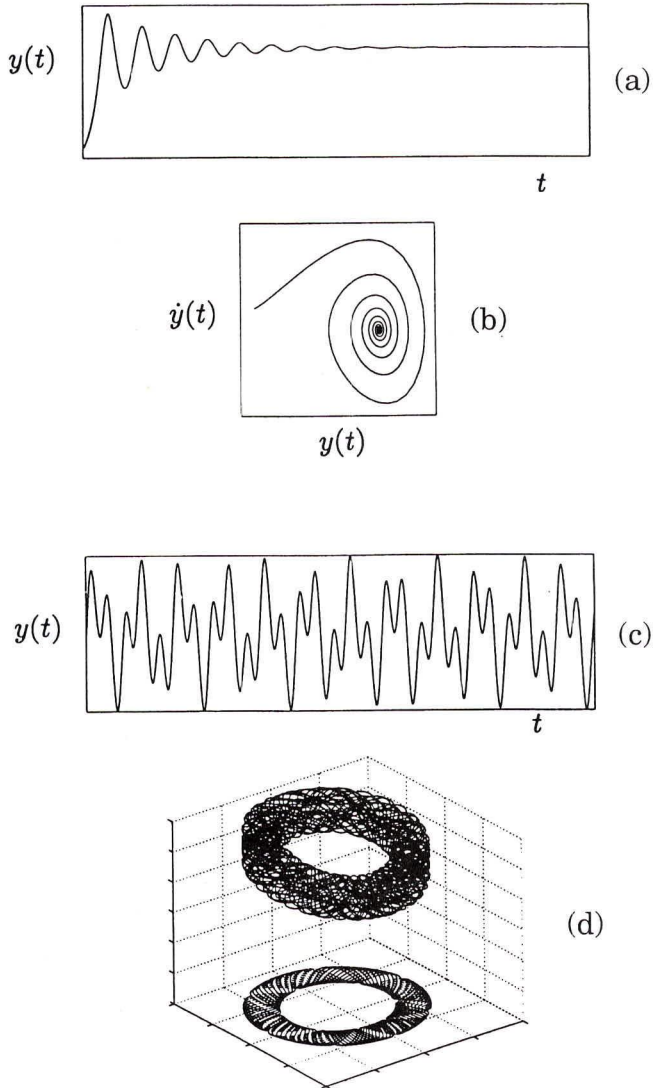
Time series and respective attractors. (a) damped oscillations settling onto a (b) point attractor. (c) quasi-periodic oscillations lie on a (d) torus in state space. Attractors with higher dimensions and more complicated shapes correspond to time series with greater complexity.

Figure 5
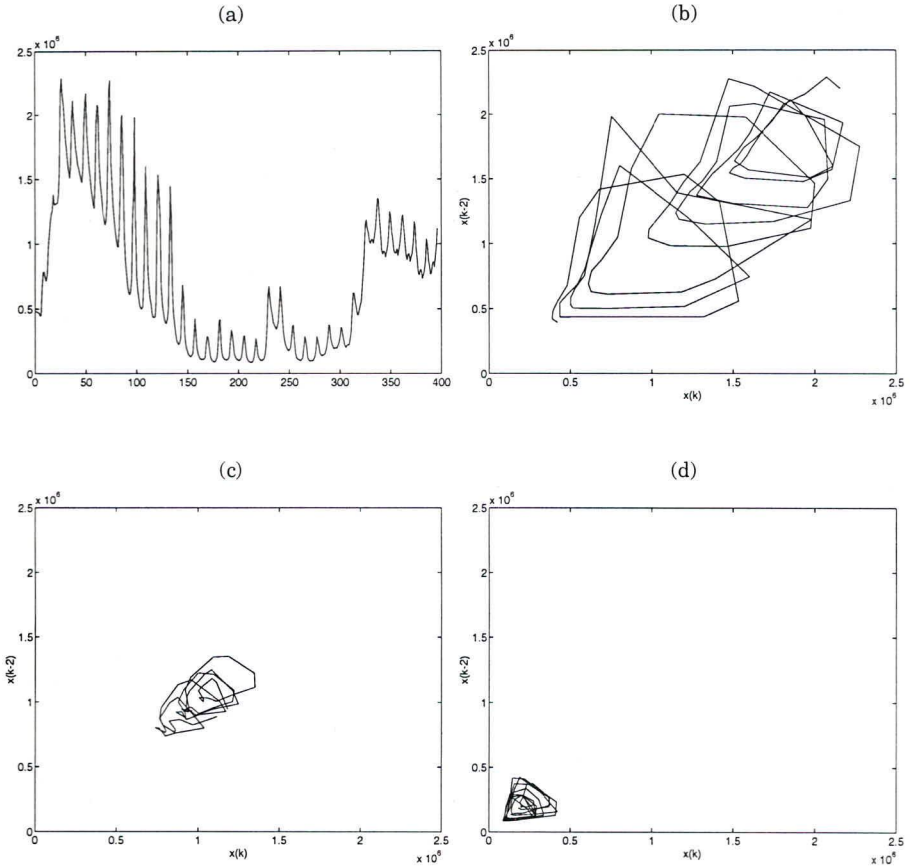
Monthly german unemplyment data. Phase-space reconstruction of this time series suggests that there are several attractors whose geometry differs. (a) Complete time series; (b), (c) and (d) different embedded attractors.

5.1 Chaotic or strange attractors

Despite some attempts, there is no widely accepted definition of chaos or chaotic attractors. In this section, a rather intuitive view of such attractors is given. Firstly, it should be realised that chaos is not a pathological dynamical regime which is only exhibited by carefully designed paradigms. Secondly, chaos is not a dynamical regime which lacks order or pattern. On the contrary, chaotic systems have well defined patterns of behaviour.

The terms *strange* and *chaotic* were coined because in the genesis of chaos, the (now well known) attractors which scientists came across were totally different from what was known at the time (thus the term *strange*), and the time series produced by such systems would not follow any predictable path (thus the term *chaotic*). Although there exist some pathological cases in which strange attractors are nonchaotic (Grebogi *et al.*, 1984), the terms 'strange' and 'chaotic' are usually used interchangeably.

It is instructive to wonder what happens if a chaotic time series is embedded in phase space. What kind of geometrical object is formed in such a space? For the sake of simplicity, assume that the dynamical system is continuous and of third order. If the time series produced by such a system is chaotic, whenever a phase space reconstruction is attempted it becomes clear that such a space cannot be a two-dimensional space. If the phase space were of dimension two, the system trajectories would cross, thus violating the uniqueness of solutions of differential equations. In order to satisfy this requirement the dimension can be increased to three and now the trajectories in phase space do not cross. The bundle of trajectories in the three-dimensional phase space forms a geometrical object which is the attractor. As mentioned before, such an attractor needs, at least, a three-dimensional space to exist but, rather surprisingly, has zero volume. This paradox cannot be solved by 'conventional' geometrical objects and a new type of object is called for. In this particular case, such an object must have a dimension which is greater than two (otherwise the attractor could exist in a two-dimensional space) but cannot have dimension three (otherwise the attractor would have zero volume.) Therefore, the object must have a dimension which is not an integer. Such objects are called *fractals* and can be thought of as the *images*, of chaos since an embedded chaotic time series will result in a fractal in the embedding space[7].

The discussion above pointed out that strange attractors are fractal objects. But what makes an attractor become fractal? To answer this

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7 There are some pathological exceptions to this rule, with which we will not be concerned here.

question in detail goes beyond the scope of this paper. However, it can be conjectured that, on the one hand the system is globally stable (since the trajectories are confined to a limited region of the phase space, that is, the attractor) and on the other hand there must be some kind of instability in the system which produces the fractal characteristics of the attractor. Thus a chaotic attractor can be thought of as being *globally stable but locally unstable.*

6 DYNAMICAL INVARIANTS

The local instability of a chaotic system is responsible for the inability of making long-term predictions of chaotic time series. This property is known as *sensitive dependence on initial conditions.* It is difficult to tell if the fractal nature of a strange attractor is responsible for the sensitivity to initial conditions or if it is the other way around. In any case, the two features usually come together in chaotic attractors and consequently a way of quantifying such attractors is to estimate how unstable and fractal they are. Measures of stability and fractality are provided by Lyapunov exponents and the correlation dimension. These concepts are briefly discussed in what follows.

6.1 Sensitivity to initial conditions

Probably the most fundamental property of chaotic systems is the sensitive dependence on initial conditions. This feature arises due to the local divergence of trajectories in state space in at least one 'direction'. This will be also addressed in the next section.

In order to illustrate sensitivity to initial conditions and one of its main consequences, it will be helpful to consider the logistic map.

$$y(k) = A\,[1\,-\,y(k\,-\,1)\,]\,y(k\,-\,1)\,. \tag{3}$$

In order to iterate equation (3) on a digital computer, an initial condition $y(0)$ is required. Using this value, the right hand side of equation (3) can be evaluated for any value of A. This produces $y(1)$ which should be 'fedback' and used as the initial condition in the following iteration. This procedure can be then repeated as many times as necessary to generate a time series $y(0), y(1), y(2), \ldots$.

A graphical way of seeing this is illustrated in Figure 6. It should be noted that the right hand side of equation (3) is a parabola, as shown in

Figure 6a. Thus to evaluate equation (3) is equivalent to finding the value on the parabola which corresponds to the initial condition. This is represented in Figure 6a by the first vertical line. The feeding back of the new value is then represented by projecting the value found on the parabola onto the 45° line. This completes one iteration.

Figure 6
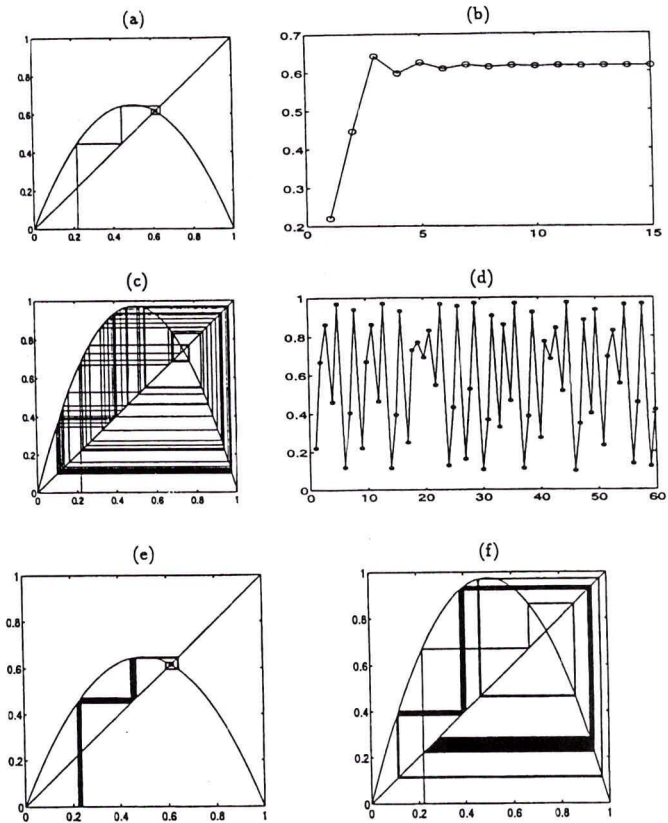
Graphical iteration of the logistic equation (3). (a) regular motion (A = 2.6) and (b) respective time series, (c) chaotic motion (A = 3.9), and (d) respective time series. In these figures the same initial condition has been used, namely y(0) = 0.22. In figures (e) and (f) an interval of initial conditions has been iterated for the same values of A as above. The intervals used were y(0) ∈ [0.22 0.24] and y(0) ∈ [0.220 0.221], respectively. Note how such an interval is amplified when the system is chaotic, (f). This is due to the sensitive dependence on initial conditions.

Chosing the initial condition $y(0) = 0.22$ and $A = 2.6$, Figure 6a shows the iterative procedure and reveals that after a few iterations the equation settles to a point attractor. The respective time series is shown in Figure 6b. The same procedure was followed for the same initial condition and $A = 3.9$. The results are shown in Figures 6c-d. Clearly, the equation does not settle onto any fixed point and not even onto a limit cycle. In fact, it is known that equation (3) displays chaos for $A = 3.9$.

What happens if instead of a single initial condition an interval of initial conditions is iterated? This is shown in Figures 6e-f. For $A = 2.6$, the map will eventually settle to the same point attractor as before. This is a typical result for regular stable systems and it illustrates how all the trajectories based on the initial conditions taken from the original interval converge to the same attractor.

Considering a much narrower interval of initial conditions and proceeding as before yielded the results shown in Figure 6f for which $A = 3.9$. Clearly, the interval of initial conditions was widened at each iteration. Such an interval can be interpreted as an error in the original initial condition, $y(0) = 0.22$. In practice errors in initial conditions will be always present due to a number of factors such as noise, digitalisation effects, round-off errors, finite wordlength precision, *etc.* It is this effect of amplifying errors in initial conditions which is known as the sensitive dependence on initial conditions and an immediate consequence of this feature is the impossibility of making long-term predictions for chaotic systems. The next section describes indices which quantify the sensitivity to initial conditions.

6.2 Lyapunov exponents

Lyapunov exponents measure the average divergence of nearby trajectories along certain 'directions' in state space. A chaotic attracting set has at least one positive Lyapunov exponent and no Lyapunov exponent of a non-chaotic attracting set can be positive. Consequently such exponents have been used as a criterion to determine if a given attracting set is or is not chaotic (Wolf, 1986). Recently the concept of *local* Lyapunov exponents has been investigated (Abarbanel, 1992). The local exponents describe orbit instabilities a fixed number of steps ahead rather than an infinite number. The (global) Lyapunov exponents of an attracting set of length N can be defined as[8]:

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8   Many authors use $\log_2$ in this definition.

$$\lambda_i = \lim_{N \to \infty} \frac{1}{N} \log_e j_i(N) \qquad i = 1, 2, ..., n , \qquad (4)$$

where $\log_e = \ln$ and the $\{j_i(N)\}_{i=1}^n$ are the absolute values of the eigenvalues of

$$[Df(y_N)] [Df(y_{N-1})] \cdots [Df(y_1)], \qquad (5)$$

where $Df(y_i) \in \mathbb{R}^{n \times n}$ is the Jacobian matrix of the n-dimensional differential equation (or discrete map) evaluated at y_i, and $\{y_k\}_{k=1}^N$ is a trajectory on the attractor. Note that n is the dynamical order of the system.

The estimation of Lyapunov exponents is known to be a nontrivial task. The simplest algorithms (Wolf *et al.*, 1985; Moon, 1987) can only reliably estimate the largest Lyapunov exponent (Vastano, Kostelich, 1986). Estimating the entire spectrum is a typically ill-conditioned problem and requires more sophisticated algorithms (Parker, Chua, 1989).

In view of such difficulties and the fact that the largest Lyapunov exponent λ_1, is in many cases the only positive exponent and that this gives an indication of how far into the future accurate predictions can be made, it seems appropriate to use λ_1 to characterise a chaotic attracting set (Rosenstein *et al.*, 1993).

6.3 Correlation dimension

Another quantitative measure of an attracting set is the fractal dimension. In theory, the fractal dimension of a chaotic (non-chaotic) attracting set is non-integer (integer). An exception to this rule are *fat fractals* which have integer fractal dimension which is consequently inadequate to describe the properties of such fractals (Farmer, 1986). Nontheless, like the largest Lyapunov exponent, the fractal dimension can, in principle, be used not only to diagnose chaos but also to provide some further dynamical information (Grassberger *et al.*, 1991). A deeper treatment can be found in (Grassberger, Procaccia, 1983a) for raw data and in (Badii *et al.*, 1988; Sauer, Yorke, 1993) for filtered time series.

The fractal dimension is related to the amount of information required to characterise a certain trajectory. If the fractal dimension of an attracting set is $D + \delta$, $D \in \mathbb{Z}^+$, where $0 < \delta < 1$, then the smallest number of first-order differential equations required to describe the data is $D + 1$.

There are several types of fractal dimension such as the *point-wise dimension, correlation dimension, information dimension, Hausdorff dimension, Lyapunov dimension*, for a comparison of some of these dimensions see (Henstschel, Procaccia, 1983; Moon, 1987). For many strange attractors, however, such measures give roughly the same value (Moon, 1987; Parker, Chua, 1989). The correlation dimension[9] (Grassberger, Procaccia, 1983b), however, is clearly the most widely used measure of fractal dimension employed in the literature.

A time series $\{y_i\}_{i=1}^{N}$ can be embedded in the phase space where it is represented as a sequence of d_e-dimensional points $y_j = [y_j \; y_{j-1} \; \dots \; y_{j-d_e+1}]$. Suppose the distance between two such points is[10] $S_{ij} = |y_i - y_j|$ then a correlation function is defined as (Grassberger, Pro-caccia, 1983b):

$$C(\varepsilon) = \lim_{N \to \infty} \frac{1}{N} \text{ (number of pairs } (i,j) \text{ with } S_{ij} < \varepsilon). \tag{6}$$

The correlation dimension is then defined as

$$D_c = \lim_{\varepsilon \to \infty} \frac{\log_e C(\varepsilon)}{\log_e \varepsilon}. \tag{7}$$

For many attractors D_c will be (roughly) constant for values of ε within a certain range. In theory, the choice of d_e does not influence the final value of D_c if d_e is greater than a certain value. In particular, it has been shown that provided there are sufficient noise-free data, $d_e = Ceil (D_c)$, where $Ceil (\cdot)$ is the smallest integer greater than or equal to D_c (Ding *et al.*, 1993). In practice, due to the lack of data and to the presence of noise, $d_e > Ceil (D_c)$, thus several estimates of the correlation dimension are obtained for increasing values of d_e. If the data were produced by a low-dimensional system, such estimates would eventually converge. Of course, these results depend largely on both the amount and quality of the data available. For a brief account of data requirements, see section 7.2 below.

Probably the greatest application of the correlation dimension is to diagnose if the underlying dynamics of a time series have been produced by a low-order system. Because this is an important problem, the estimation

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9   This measure can be seen as a *generalised dimension*, and is considered to be the easiest to estimate reliably (Grasbergber, 1986b) and thus remains the most popular procedure so far.

10 Several norms can be used here such as Euclidean, l_1, *etc.*

of correlation dimension has attracted much attention in recent years as will be seen in the next section.

7 DIAGNOSING NONLINEARITIES AND CHAOS

One of the first steps in time series analysis is to verify if the data at hand are nonlinear and, if so, if they are chaotic. The outcome of this first step will 'set the atmosphere' in the later stages. Of course, if there is no evidence for nonlinearity in the data, simple linear and (perhaps) stochastic models should be considered. On the other hand, if the data seem to be nonlinear then nonlinear model representations should be tried, or in the words of Brock and Sayers (1988):

> "If you have strong evidence from our methods that nonlinear structure exists, then it is worth spending resources trying to identify and estimate it".

7.1 Diagnosing chaos

In general, the problem of diagnosing chaos can be reduced to estimating invariants which would suggest that the data are chaotic. For instance, positive Lyapunov exponents and non-integer dimensions would suggest the presence of chaos. The main question is how to confidently estimate such properties from the data, especially when the available records are relatively short and possibly noisy.

In view of the relevance of this topic, a number of different techniques have appeared in recent literature. In order to facilitate the discussion, the procedures used in diagnosing chaos can be divided in three major groups as shown below. Needless to say, such division is not an agreed upon standard, nor is the list of methods exhaustive.

Non-parametric methods

These include the use of tools which take the data and estimate dynamical invariants which, in turn, will give an indication of the presence of chaos. Such tools include power spectra, the largest Lyapunov exponent, the correlation dimension, reconstructed trajectories, Poincaré sections, relative rotation rates *etc*. Detailed description and application of these techniques can be found in the literature (Moon, 1987; Tufillaro, *et al.*, 1990; Denton, Diamond, 1991). For a recent comment of the practical difficulties in using Lyapunov exponents and dimensions for diagnosing chaos see (Mitschke, Dämmig, 1993).

Two practical difficulties common to most of these approaches are the number of data points available and the noise present in the data. These aspects are briefly discussed in section 7.2.

Prediction-based techniques

Some methods try to diagnose chaos in a data set based upon prediction errors (Sugihara, May, 1990; Casdagli, 1991; Elsner, 1992; Kennel, Isabelle, 1992). Thus predictors are estimated from, say, the first half of the data records and used to predict over the last half. Chaos can be, at least in principle, diagnosed based on how the prediction errors behave as the prediction time is increased (Sugihara, May, 1990), or based on how the prediction errors related to the true data compare to the prediction errors obtained from 'faked' data which are random but have the same length and spectral magnitude as the original data (Kennel, Isabelle, 1992). A related approach has been termed *the method of surrogate data* (Theiler *et al.*, 1992).

Regardless of which criterion is used to decide if the data are chaotic or not, predictions have to be made. Clearly, the viability of these approaches depends on how easily predictors can be estimated and on the convenience of making predictions. Once a predictor is estimated, criteria and statistics such as the ones presented in (Sugihara, May, 1990; Kennel, Isabelle, 1992) can be used to diagnose chaos.

Methods related to economic applications

Clearly, most of the methods cited in the two groups above can be applied to economic time series. This group, however, includes a few techniques which seem to have been first developed for applications in economics. Brock (1986) suggested a test which compares the correlation dimension and the largest Lyapunov exponent of a time series and of the residuals produced by fitting a linear model to the set of data. A good discussion of the application of Lyapunov exponents and correlation dimension estimates applied to economic data can be found in (Brock, Sayers, 1988). In particular, these authors pointed out that *"near unit root processes can generate low dimension estimates and apparently positive largest Lyapunov exponent estimates"*.

Some alternative tests used to distinguish high-dimensional chaos, *i.e.* 'randomness', from low-dimensional chaos, *i.e.* deterministic and complex dynamics, are based on variants of the correlation dimension. The so-called BDS test uses a statistic defined in terms of the correlation dimension and can be used to test the null hypothesis of identically independent distribution (IID) (Brock *et al.*, 1988). A statistic based on the *lagged correlation integral* has been defined to test for statistical independence under the assumptions of stationarity and gaussianity (Dechert, 1989).

These approaches have been reviewed in (Brock, Dechert, 1991). For an earlier comparison of several methods see (Barnett, Chen, 1988).

7.2 Data requirements

The length and quality of the data records are crucial in the problem of characterisation of strange attractors. At present, there seems to be no general rule which determines the amount of data required to learn the dynamics, to estimate Lyapunov exponents and the correlation dimension of attractors. However it is known that

> *"in general the detailed diagnosis of chaotic dynamical systems requires long time series of high quality"* (Ruelle, 1987).

It should be pointed out that this is a very rapidly developing research area and significant improvements have been made. Consequently, new algorithms tend to require less data. On the other hand, there are several limitations which are not directly imposed by computational procedures but rather are inherent in systems with complex dynamics. Thus the information presented below should be seen in the light of these remarks.

It has been argued that the data required to estimate the Lyapunov exponents should satisfy $N > 10^D$ [quoted *in* Rosenstein *et al.* (1993)] and $N > 30^D$ where D is the dimension of the system (Wolf *et al.*, 1985).

Fairly long time series are also required for estimating the correlation dimension. In fact, it has been pointed out that dimension calculations generally require larger data records than those needed to estimate Lyapunov exponents (Wolf, Bessoir, 1991). For a strange attractor, if insufficient data is used the results would indicate the dimension of certain parts of the attractor rather than the dimension of the entire attractor (Denton, Diamond, 1991). However, results have been reported which suggest that consistent estimates of the correlation dimension can be obtained from data sequences with less than 1000 points (Abraham *et al.*, 1986). On the other hand, there seems to be evidence that "spuriously small dimension estimates can be obtained from using too few, too finely sampled and too highly smoothed data" (Grassberger, 1986a). Moreover, the use of short series and noisy data sets may cause the correct scaling regions to become increasingly shorter and may cause the estimate of the correlation dimension to converge to the correct result for relatively large values of the embedding dimension (Ding *et al.*, 1993). Thus there seems to be no agreed upon rule to determine the amount of data required to estimate dimensions with confi-

dence but it appears that at least a few thousand points for low dimensional attractors are needed (Essex, Nerenberg, 1991). In particular, $N \geq 10^{D_c/2}$ has been quoted in (Ding *et al.*, 1993).

One result that claims to be the first case of identified chaos in economic data is that of Barnett and Chen who report about a

> "*research project that has been successful in identi-fying economic chaotic attractors from long time series of unusually high quality data. In particular, those attractors can explain the dynamical behavior of the broad Divisia monetary aggregates and hence can reveal information about the nature of the dy-namical system that produces the observed moneta-ry services path over time... Although there previously have been a number of successful appli-cations of deterministic chaotic dynamics in econo-mic theory, we believe that our results represent the first clearly successful empirical application*" (Bar-nett, Chen, 1988).

The length and quality of the data set depends on a number of factors. However, it seems instructive to indicate the length of real economic time series which have been used in the literature in connection with testing for nonlinearities and chaos. 5200 daily stock returns (Scheinkman, LeBa-ron, 1989). Two 468 (monthly) samples of US base money and interest rates on three month US Treasury bills (Granger, Hallman, 1991). Approximately 150 sample observations were used to test for low dimensional dynamics in the post War II, US quarterly GNP. However, it was warned that there seemed not to be enough information in such a short time series to validate the results (Brock, 1986). Moreover, in connection with other short time series it has been warned that

> "*low dimension and 'rough' estimates (even though apparently positive) of Lyapunov exponents do not make the case for low-dimensional deterministic chaos in data sets as short as ours*" (Brock, Sayers, 1988).

It has been concluded, based on Monte Carlo simulations, that the BDS statistic does not approximate asymptotic normality for sample sizes under 500 (Brown *et al.*, 1991).

7.3 Diagnosing nonlinearities

The remarks above make it plainly clear that very few, if any, economic time series would qualify as good candidates from which reliable dynamical invariants could be estimated. It is fundamental to realise that the difficulties in obtaining long time series go well beyond problems such as storage and computation time. In fact, the greatest difficulty is *not* even the impractically long span of time which may be required to record thousands of, say, monthly observations. What seems to be the bottleneck of the whole procedure is the system itself, because a basic assumption made by most algorithms is that all the data are *on a single attractor*. This is equivalent to requiring that the data should be stationary. Therefore if a system is evolving from one attractor to another[11], and the time spent on a 'single' attractor is insufficient to characterise such a dynamical regime, the chances of confidently estimating dynamical invariants from such data with the algorithms developed so far are very slim. An important problem related to this is to determine periods of time (windows of data) within which the system can be considered stationary. A test for stationarity has been recently presented in the context of nonlinear dynamics (Isliker, Kurths, 1993).

The previous discussion suggests that in many practical situations, especially concerning economic time series, it might be more realistic (and also easier) to diagnose nonlinearities in the data rather than trying to verify if the system has positive Lyapunov exponents or if its correlation dimension is non-integer.

The so-called W test can detect nonlinearities by using the residuals of a linear model (Brock, Sayers, 1988). This method has been used to test for the presence of hysteretic path dependence of monthly US aggregates in the period 1971-1993 (Peel, Speight, 1994). The correlation dimension has been used to detect nonlinear departures from random-walk behavior in stock returns (Scheinkman, LeBaron, 1989). These and other related approaches have been discussed in (Barnett *et al.*, 1992).

In some applications, it might be of interest to diagnose other dynamical features apart from chaos or even nonlinearities. In (Granger, Hallman, 1991) a test for linear and nonlinear cointegration was applied to two sets of real data. The procedure was based on a unit root test, and evidence has been found for nonlinear cointegration suggesting the existence of a nonlinear attractor.

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11 This seems to be the case with the German unemployment data, see Figure 5. Here the problem of amount of data would be critical since these assumed attractors have around 100 sample observations each.

It seems appropriate to conclude this section with a caveat:

> " ... *our current results suggest that there are enough nearly arbitrary degrees of freedom available to economic researchers to permit them to find whatever they may wish to find*" (Barnett *et al.*, 1992).

8 FINAL REMARKS AND FURTHER READING

This paper has briefly touched upon some aspects of the theory and practice of nonlinear dynamics which seem relevant for applications in economics. None of the discussions and definitions should be considered exhaustive nor mathematically precise. This was the price the authors deliberately chose to pay in order to produce what they hope will turn out to be a readable, formulae-free and clear introduction to the subject.

Another objective which was kept in mind throughout the paper was to provide relevant references for further reading. Although it would be impossible to compile a complete list, a few more works are cited in what follows.

The following references seem to be a good starting point. The books (Gleik, 1987) and (Stewart, 1989) are a good introduction for the average reader. A more formal coverage is given by (Thompson, Stewart, 1986) and (Moon, 1987). For a mathematical exposition on the subject see (Guckenheimer, Holmes, 1983) and (Wiggins, 1990). A good account on computer algorithms for nonlinear systems applications can be found in (Parker, Chua, 1989). Good surveys on modelling and analysis of chaotic series can be found in (Grassberger *et al.*, 1991; Abarbanel *et al.*, 1993).

Some specific information about the wide application in economics of the ideas and techniques reviewed in this paper are mentioned below.

In the first place, special issues of three journals that were devoted entirely to this subject are:

i) the first issue of the 40th volume of the *Journal of Economic Theory*, published in October 1986;

ii) the supplement to volume 7 of the *Journal of Applied Econometrics*, (Special Issue – Nonlinear Dynamics and Econometrics), December 1992;

iii) the 5th number of the 25th volume of the *International Journal of Systems Science*, (Special Issue – Monetarism versus Keynesianism), which appeared in May 1994.

Jess Benhabib collected in a single volume a set of 21 papers that provide an ample overview of the various economic mechanisms that produce cyclic or chaotic dynamics in equilibrium (Benhabib, 1992). These contributions to the literature discuss the subject of oscillatory equilibria in models with overlapping generations as well as those of the Ramsey type with infinitely lived representative agents.

Boldrin and Woodford provide a survey of the literature dealing with endogenous cycles (Boldrin, Woodford, 1992). These authors use rigorously formulated equilibrium models, in which agents optimise with perfect foresight, to show that endogenous fluctuations (either periodic or chaotic) can persist in the absence of exogenous shocks. Some books already cited in former sections of this article also include comprehensive surveys on the subject (Rosser, 1991; Anderson *et al.*, 1988).

Within the Keynesian framework there are several contributions that attempt to explain how complicated dynamics can be generated (Torre, 1977; Dana, Malgrange, 1984). These authors show that, by varying a bifurcation parameter, different regimes can be obtained from a given model ranging from an attracting steady state to an apparently chaotic state. More on bifurcations can be found in the following books (Guckenheimer, Holmes, 1983; Wiggins, 1990). A nonlinear version of Samuelson's multiplier-accelerator model is presented by (Day, Shafer, 1985).

Following the tradition of the descriptive growth models initiated by Solow in the 1950s other authors used alternative hypotheses to show that those changes could bring about chaotic dynamics (Day, 1982; Bhaduri, Harris, 1987; Stutzer, 1980).

Another type of models that also present endogenous cycles are those based on the concept of overlapping generations. In these models people live two periods: in the first period they save part of their incomes to spend in the next, that is, we have life-cycle savings by individuals. Benhabib and Day studied model economies with these characteristics from the point of view of nonlinear dynamics under different sets of assumptions to obtain chaos (Benhabib, Day, 1980; Benhabib, Day, 1982). Other studies have generalized and improved these results on overlapping generations economies (Grandmont, 1985; Farmer, 1986; Reichlin, 1986).

Opposite to the overlapping generations models are the Ramsey type models. In these competitive model economies individuals live forever, have perfect foresight, and try to maximise the discounted sum of their

utilities over the infinite horizon. It was shown that chaotic dynamics can be generated by optimal solutions of infinite horizon growth models of this kind (Boldrin, Montrucchio, 1986; Deneckere, Pelikan, 1986).

A variety of other economic models have been found to exhibit chaotic dynamics under reasonable hypotheses and parameter values. One of them considers the theory of the firm (Albin, 1987), other consumer behaviour (Benhabib, Day, 1981), other proves chaos in a simple model of research and development (Baumol, Wolf, 1983). Trying to provide a *"sketch of the wealth of current research areas"*, the following list of applications classified by type of research area has been provided in (Nijkamp, Reggiani, 1995):

i) economic growth theory, with emphasis on business cycles (Balducci *et al.*, 1984; Brock, Sayers, 1988; Day, 1982; Funke, 1987; Grandmont, 1985; Hommes, 1991; Puu, 1989);

ii) theory of structural economic change, with emphasis on the emergence and existence of long waves (Nijkamp, 1987; Rasmussen *et al.*, 1985; Sterman, 1988);

iii) innovation theory, with emphasis on R&D behaviour (Baumol, Wolf, 1983; Nijkamp *et al.*, 1991);

iv) theory of economic competition, with emphasis on limited competition and game theory (Albin, 1987; Dana, Montrucchio, 1986; Deneckere, Pelikan, 1986; Ricci, 1995);

v) theory of economic equilibrium, with emphasis on growth and trade (Hommes, Nusse, 1989; Lorenz, 1987; Nusse, Hommes, 1990).

The spirit of this paper has been to point out that a typical economic time series or economic system can (and perhaps should) be seen as a dynamical system and analysed as such. A key concept to achieve this is the *embedding* of the real data in a reconstruction space. This is the basis upon which a lot of what is known as *chaos theory* rests. There are nontrivial results which guarantee that if the embedding is successful, it is possible to analyse the *embedded* data and in so doing infer dynamical properties of the *real* system. We close this paper with this key concept and recall that Figure 3 summarises this pictorially.

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